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Globally non-causal space–times

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Abstract. It is shown that if a space–time has an isometry group whose Killing vector is time-like in a region V and if there exists one closed time-like curve somewhere in the space–time, then under suitable conditions on V (including the absence of horizons) there is a closed time-like curve through every point of the space–time.

The possible existence of a smooth closed time-like curve in the vicinity of a naked singularity (Tipler 1976) leads to the awkward possibility that the space–time as a whole is a chronology violating set.

The possibility that this was not the case in the sense that no closed paths (whether geodesics or not) existed through points of infinity was investigated in the Kerr (de Felice and Calvani 1979) and Kerr–Newman metrics (de Felice 1981) but then explicit examples were found of these curves showing that in these cases causality violation is a non-avoidable pathology, provided that $M^2 < a^2 + Q^2$.

In this paper we shall prove a general theorem which states that a space–time with a stationary 'outer region' V, which admits one closed time-like curve somewhere (for instance, arising from a naked singularity), has the whole space as a chronology violating set.

Theorem. Given a space–time (M, g) with a smooth one-parameter group of isometries $\{\phi_t: M \to M, \forall t \in \mathbb{R}\}$ with $\boldsymbol{\xi}$ the generator of the group (a Killing vector), suppose there exists a connected open set $V \subset M$ with the properties

(i) $\phi_t(V) = V \; \forall t \in \mathbb{R}$,

(ii) ξ is time-like in V,

(iii) $(V, g|V)$ has a partial cauchy surface S,

(iv) the space–time has no horizons relative to V: that is, for all $x \in M$, $\exists p, q \in V$ such that $p \ll x \ll q$,

(v) there exists a smooth closed time-like curve in M.

Then for any $x \in M$, there exists a closed time-like curve through x.

In order to prove the theorem, let us prove the following lemmas.

Lemma 1.

$$V = \bigcup_{t \in \mathbb{R}} \phi_t(S).$$

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Suppose that $W := \bigcup_{t\in\mathbb{R}} \phi_t(S)$. Then, by (i), $W \subset V$; to prove lemma 1 we have to show that $V - W = \phi$. Suppose it were not; then there would exist a

$$x \in \bar{W} \cap \overline{V - W}. \tag{1}$$

The curve $\gamma : t \to \phi_t(x)$ has a time-like tangent vector ξ so we can choose t_1 and t_2 and a normal neighbourhood N of x such that

$$Q := J^+(z_2, N) \cap J^-(z_1, N)$$

is non-empty with $z_2 \ll_N x \ll_N z_1$ (where $z_i = \phi_{t_i}(x)$). Call K the connected component of γ in the set of the future time-like curves which connect z_2 to z_1, with the compact open topology.

Since x belongs to the closure of W, there is a sequence of points which tends to $x : (y_n)_{n=1}^{\infty} \to x$ with $y_n \in W$. For some n_0, $y_{n_0} \in \operatorname{int} Q$; let t_0 be such that $y_{n_0} \in \phi_{t_0}(S) = S'$, say, and let K' be the subset of K consisting of those curves which meet S'. Because y_{n_0} belongs to at least one curve of K', K' is non-empty; also K' is not K because some reparametrisation of γ lies in $K - K'$. Set $h \in (K - K') \cap \bar{K}'$, which is non-empty since K is connected and K' open; then there exists a sequence $(k'_n)_{n=1}^{\infty}$ of curves of K' tending to h. Suppose that k'_n meets S' in w_n, and that $(w_n)_{n=1}^{\infty} \to w \in Q$. Now it is easy to show that $w \neq z_1$; in fact we have $y_{n_0} \ll_N z_1$, so that if w were z_1, then there would exist a w_{n_1} with the property: $y_{n_0} \ll_N w_{n_1}$; but w_{n_1} belongs to S' and that contradicts the causality of S' which is implied by (iii). Similarly $w \neq z_2$. Moreover, w lies on h; thus $z_2 \ll_N w \ll_N z_1$. For any neighbourhood U of w we can find an h' so close to h that there are points p and q on h' such that $p \ll_N w \ll_N q$, but $h' \in K - K'$ so $w \in \operatorname{edge}(S')$ (Hawking and Ellis 1973, p 202). But $\operatorname{edge}(S') = \phi$ by (iii) and that contradicts (1). This proves lemma 1. □

Lemma 2. For any $x, y \in V$, $\exists t$ such that

$$x \ll_V \phi_t(y).$$

Since V is connected, open and M is locally path-connected, V is also path-connected. Thus there exists a map $\rho : [0, 1] \to V$ such that $\rho(0) = x$, $\rho(1) = y$: further, it is always possible to take ρ to be smooth.

From lemma 1, for all s there exists $\tau(s)$ such that

$$\phi_{\tau(s)}\rho(s) \in S.$$

Let α be a positive number, to be fixed shortly, and let

$$\rho'(s) := \phi_{\alpha[\tau(s)-\tau(0)]}\rho(s).$$

This has the property that $\rho'(0) = x$, $\rho'(1) = \phi_t(y)$ where $t = \alpha\ [\tau(1) - \tau(0)]$. The map ρ' is clearly smooth and is such that the corresponding tangent vector is

$$\dot{\rho}' = \alpha\xi + \phi_{\alpha[\tau(s)-\tau(0)]^*}\dot{\rho}(s).$$

Since the range of ρ' is compact and ξ is time-like we can choose α large enough that $\dot{\rho}'$ is time-like. This implies that x and $\phi_1(y)$ are connected by a time-like curve, proving the lemma. □

Lemma 3. For any points $x, x' \in V$ we have $x \ll x'$. Suppose $\gamma : [0, 1] \to M$ is the smooth closed time-like curve premised in the theorem. Set $u := \gamma(0) = \gamma(1)$. Let

$p, q \in V$ be such that

$$p \ll u \ll q$$

according to (iv). From lemma 2, we can find a t_1 such that

$$x \ll \phi_{t_1}(p) \ll \phi_{t_1}(u) \ll \phi_{t_1}(q). \qquad (2)$$

Moreover again using lemma 2, we can find t_2 so that

$$\phi_{t_1}(q) \ll \phi_{t_2}(x')$$

which implies, using (2),

$$\phi_{t_1-t_2}(u) \ll \phi_{t_1-t_2}(q) \ll x'. \qquad (3)$$

The lemma will be proved if, comparing (2) with (3), we can show that

$$\phi_{t_1}(u) \ll \phi_{t_1-t_2}(u). \qquad (4)$$

Now define $\gamma'(s) = \phi_{t_1-st_2}[\gamma(ns)]$ for some positive integer n. Then

$$\dot{\gamma}'(s) = -t_2\xi + n\phi_{t_1-st_2}^{*}\dot{\gamma}.$$

$\dot{\gamma}'$ is future time-like and the domain γ' is compact, so for large enough $n > 0$, $\dot{\gamma}'$ is future time-like. Hence (3) is verified by γ' and the lemma is proved. □

We are now in a position to prove the theorem. Let x be any point in M. By (iv), we can find $p, q \in V$ such that $p \ll x \ll q$. By lemma 3, $q \ll p$, and so $x \ll q \ll p \ll x$; i.e. there is a closed time-like curve through x. □

If a naked singularity ever formed as a result of a gravitational collapse, it is plausible (bearing in mind the Kerr metric) that its gravitational field would be described by a space–time which does satisfy properties (i)–(v). In this case the theorem we have proved shows quite generally that the space–time would be globally non-causal. This means that an observer at a point x could receive information from his own future.

Unless a suitable alternative is available it seems we have to face the hard but also challenging perspective of accepting these extreme consequences of general relativity as a real possibility in nature.

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